Acoustical Klein-Gordon Equation: A Time-Independent Perturbation Analysis

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The perturbation analysis of an ideal acoustical duct was first made by Rayleigh in 1878 and the result has since stood in the literature. However, the analysis is based on the assumption of potential and kinetic energy densities that remain constant as a change in cross section occurs, whereas, in fact, they may fluctuate significantly in comparison to the slowly varying "wave function," $\Psi(x, t)$, of the acoustical Klein-Gordon equation. The square of the time-independent eigenfunction, $\psi^2(x)$, is directly proportional to the potential energy per unit length of fluid, and it is shown that it is precisely the perturbation in potential energy that defines correctly the eigenvalue shifts.

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Introduction.—The effects of a small perturbation, $\delta S(x)$, on the eigenvalues of a pipe of otherwise uniform cross section, S_0, were first examined by Rayleigh [1] in 1878, and the result has since stood in the literature. For a pipe of nominal length l closed at the input, $x = 0$, and open at $x = l$, for example, the effective end correction at the nth mode, Δl_n, can be written as

$$\Delta l_n = \int_0^l \cos\left[\frac{(2n-1)\pi x}{l}\right]\frac{\delta S(x)}{S_0}\,dx, \quad n = 1, 2, 3, \ldots, \tag{1}$$

so that a constriction at a pressure antinode raises a resonance, and an expansion lowers it, the converse holding at a node.

However, such analyses (see also Chiba and Katiyama [2], Schroeder [3], Fant [4], and Stevens [5]) are based on the assumption of a fluid displacement and potential and kinetic energy densities that remain constant as the change in cross section occurs, when in fact they may fluctuate significantly. Now, by applying the methods of wave mechanics to the analysis of acoustic perturbations for the first time, it can be shown that it is precisely the change in potential energy that defines correctly the eigenvalue shifts.

Theory.—It was Salmon [6] who first noted that, for a progressive plane wave in an ideal fluid, the excess pressure, $p'(x, t)$, and area function, $S(x)$, of the Webster equation [7], must together obey the principle of conservation of energy such that, averaged over a period, τ,

$$\langle p'^2(x, t)\rangle_\tau S(x) = \text{const.} \tag{2}$$

Whereas the excess pressure undergoes large fluctuations with change in cross section, Eq. (2) suggests the definition of a slowly varying "acoustical wave function," $\Psi(x, t)$, as

$$\Psi(x, t) = p'(x, t)[S(x)]^{1/2}, \tag{3}$$

where $|\Psi(x, t)|^2$ is directly proportional to the potential energy per unit length of fluid. Substitutions within the Webster equation then result in the reduced [6,8] or "Klein-Gordon" [9,10] form

$$\frac{\partial^2\Psi(x, t)}{\partial t^2} = c^2\left\{\frac{\partial^2\Psi(x, t)}{\partial x^2} - U(x)\Psi(x, t)\right\}, \tag{4}$$

where c is the speed of sound in the ambient medium. The "acoustical potential function," $U(x)$, is defined as

$$U(x) = \frac{d^2[S(x)]^{1/2}/dx^2}{[S(x)]^{1/2}}, \tag{5}$$

and previous authors have noted that the time-independent equation,

$$\frac{d^2\psi(x)}{dx^2} + [k^2 - U(x)]\psi(x) = 0, \tag{6}$$

for k the free space wave number, is analogous to the steady-state Schrödinger equation. Setting $\hat{k} = [k^2 - U(x)]^{1/2}$, then for $d\hat{k}/dx \ll \hat{k}$, Eq. (6) has eigenfunction solutions, $\psi(x)$, in terms of a dispersive wave number, $\hat{k}$, namely,

$$\psi(x) = Ae^{-i\hat{k}x} + Be^{i\hat{k}x} \tag{7}$$

[10]. Such dispersive solutions elucidate significant variations in phase velocity from predictable phenomena, and previous work [9] has considered in detail those due to piecewise constant potential functions, for which $U(x) = U_0$. The case $U_0 > 0$ describes the acoustical barrier, corresponding to a region of positive curvature, whereas that of $U_0 < 0$ describes the acoustical well, corresponding to a negative curvature.

Perturbation theory: constant or slowly varying perturbation.—Consider an ideal uniform duct closed at the

input, $x = 0$, and terminated at $x = l$ in a lossless radiation impedance, $Z_{\text{rad}} = 0$. Let us assume a constant, perturbing potential function, $U(x) = U_0, 0 < x < l$, corresponding to the imposition of a small "scaled curvature" on the uniform bore, according to (5) [9]. The dispersion defines corrected eigenfunctions, $\psi_{cn}(x)$, as

$$\psi_{cn}(x) = A_n \cos(\hat{k}_n x), \tag{8}$$

and for nominal eigenvalues, $k_n = (2n-1)\pi/(2l)$, determines corrected eigenvalues, k_{cn}, as

$$k_{cn} = \left\{\left[\frac{(2n-1)\pi}{2l}\right]^2 + U_0\right\}^{1/2}, \qquad n = 1, 2, 3, \ldots. \tag{9}$$

It can immediately be seen that a wave-mechanical barrier perturbation will raise an eigenvalue, a well lower it, and that the percentage shifts are larger at low frequencies, for which $k_n^2 \ll U_0$.

The "conserved" wave-mechanical formalism also allows an insightful application of Ehrenfest's theorem [3,11] to the calculation of resonance shifts.

Defining the perturbation as δk_n, where $\delta k_n = \hat{k}_n - k_n$, the eigenfunctions (8) may be rearranged as

$$\psi_{cn}(x) = A_n \cos\{[k_n + \delta k_n]x\}. \tag{10}$$

For small $\delta k_n x$, the corrected total potential energy, E_{cpn}, may be expanded to first order as

$$E_{cpn} = \frac{A_n^2}{4\rho_0 c^2}\int_0^l 1 + \cos\left[\frac{(2n-1)\pi x}{l}\right] - \alpha_n(x)dx, \tag{11}$$

where

$$\alpha_n(x) = 2\delta k_n x \sin\left[\frac{(2n-1)\pi x}{l}\right], \tag{12}$$

where ρ_0 is the equilibrium density of air and $\cos[(2n-1)\pi x/l]$ is proportional to the radiation pressure, $P_n(x)$, which is the force exerted by the standing wave per unit cross-sectional area [3].

Maintaining the assumption of a standing wave so that, for the total kinetic energy, E_{kn}, then $E_{kn} = E_{pn}$, an expression for the first-order perturbation, δE_n, to the total energy of the nth eigenfunction is found simply as

$$\delta E_n = -\frac{A_n^2}{2\rho_0 c^2}\int_0^l \alpha_n(x)dx. \tag{13}$$

Now applying Ehrenfest's theorem for a small adiabatic perturbation of an undamped linear oscillator, which states that the relative shift in frequency of the oscillator is equal to the relative shift in total energy, that is

$$\frac{\delta F_n}{F_n} = \frac{\delta E_n}{E_n}, \tag{14}$$

the expression for the first-order shift in an eigenfrequency (see also [5]) is found to be

$$\frac{\delta F_n}{F_n} = -\frac{1}{l}\int_0^l \alpha_n(x)dx, \tag{15}$$

leading immediately to the result

$$\frac{\delta F_n}{F_n} = -\frac{\delta k_n}{k_n}. \tag{16}$$

Since δk_n is positive for propagation above a well but negative above a barrier, it is evident (i), from (11), that the perturbation term, $\alpha_n(x)$, may be in or out of phase with the radiation pressure, thus tending to strengthen or weaken (the amplitude of) the resonance, respectively; and (ii), from (12) and (15), that a perturbing potential function will tend to raise, lower, or have no effect on an eigenvalue, depending on the interaction of the dispersion with the phase of the sinusoidal term. It follows that constructive perturbations and potential-energy raising will be found for a barrier in the eighth of a period following an antinode, or a well in the eighth preceding it. Constructive perturbations and potential-energy lowering will be found for a well in the eighth of a period preceding a node, or for a barrier in the eighth following it. Such phase-sensitive effects are somewhat in analogy with others noted by Rayleigh, *albeit* regarding the transmission of heat.

These predictions may be validated by reference to the numerical example of a duct of length $l = 17.5$ cm, for which the nominal "quarter-wavelength" eigenvalues correspond to frequencies, F_n, of 500, 1500, 2500,..., Hz. Setting a small, constant, perturbation $|\delta k_n| = 1\ \text{m}^{-1}$ entails $|U_0| \approx 20\ \text{m}^{-2}$ at the first eigenfunction. For $U_0 = 20\ \text{m}^{-2}$, Eq. (9) yields exact shifts of +58.6, +20.6, and +12.4 Hz at the first three eigenvalues, respectively, whereas the first-order perturbation theory (16) predicts shifts of +66.5, +20.8, and +12.4 Hz. The agreement is around 86% at the first eigenfunction, rising above 99% as the Ehrenfest and trigonometric approximations become more appropriate.

For a well, setting a perturbation $U_0 = -20\ \text{m}^{-2}$ leads, by Eq. (9), to exact shifts of −66.5, −20.8, and −12.4 Hz, respectively, whereas the Ehrenfest theorem (16) predicts shifts of −58.6, −20.6, and −12.4 Hz. The agreement is 88% at the first eigenfunction, again rising with frequency.

Piecewise-constant perturbation.—Consider a narrow, rectangular, potential function barrier located at variable position, x, within the duct, of approximately Dirac form, $\delta(x)$. Although the potential-function perturbation is discontinuous and strictly localized, the effects on the area function are long range and, by Eq. (5), it remains continuous throughout [9]. In this case, due to the discontinuities in $U(x)$, the methods of the previous section cannot be used to predict the resonance shifts. However,

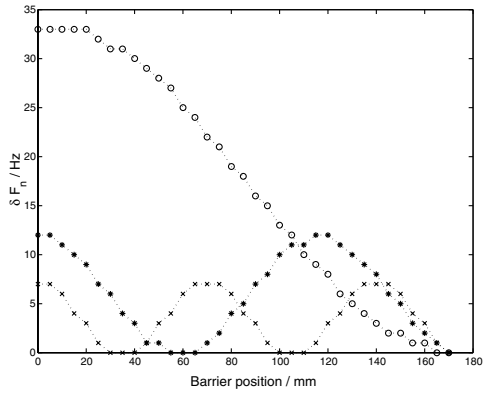

FIG. 1. Frequency shifts, δF_n, due to a barrier of form $\delta(x)$: $n = 1$ (○); $n = 2$ (∗); $n = 3$ (×).

the wave-mechanical Green's function (impulse response) solution [9] at a high-impedance source, $G_f(0|0|\omega)$, is directly proportional to the input impedance, Z_0, and is found to be

$$G_f(0|0|\omega) = \frac{\rho_0 c}{[S(0)]^{1/2}} \frac{1 + R(x, \omega)}{1 - R(x, \omega)}, \qquad (17)$$

where $\omega = ck$ and $R(x, \omega)$ is the reflection coefficient due to both the ideal termination and barrier. Equation (17) can thus be used to calculate the resonance shifts as a function of perturbation position. Figure 1 illustrates the effect on the nominal resonances, k_n, of moving the barrier in steps of 5 mm along the duct. The results were obtained at a resolution of 1 Hz, accounting for the slight lack of smoothness in the numerical solutions. The cyclic variations in frequency shift identically follow the standing wave patterns—in fact, Eq. (17) can be fitted by a term of type $\coth[k_n(x + l)]$—with a perturbing barrier at an antinode producing the maximal shift, and one at a node producing no shift. Furthermore, at $x = l/2$, halfway between node and antinode for all eigenfunctions, the entire eigenvalue spectrum is shifted together by ~50% of the maxima. Near the common antinode, the absolute frequency shift due to any given perturbation falls with increasing eigenfunction order, but this trend is reversed towards the common node. Nevertheless, the percentage shift is always greater for the lower eigenfunctions, as expected.

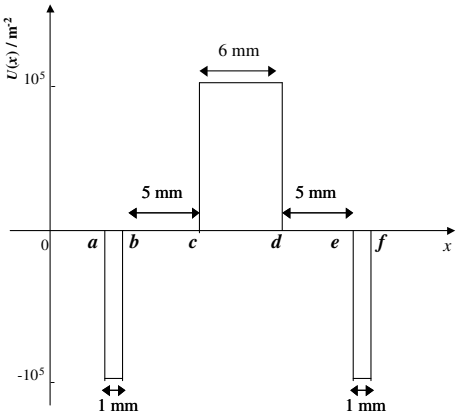

FIG. 2. Compound, symmetric, potential function.

Compound perturbations.—In this section, it will be shown that an accurate account of the effect of localized pipe expansions and constrictions on resonance involves the analysis of both positive and negative wall curvatures and elucidates perturbation phenomena specific to "compound" sequences of barriers and wells.

Consider a compound, symmetric, potential function configuration consisting of two 1 mm wide wells separated by 5 mm from a 6 mm wide barrier, $|U_0| \sim 10^5$ m^{-2}, as illustrated in Fig. 2. For an initially uniform duct and setting $S(0) = 5$ cm^2, the configuration maps to the localized constrictions of Fig. 3, approximating a discontinuous change in area function, $\delta S(x)$, of around -4.5 cm^2 and reducing the cross section to around the bottom limit for ideal and adiabatic propagation [9]. As noted in the previous section, true discontinuities of the type assumed by Rayleigh cannot be treated within the Klein-Gordon framework since, from Eq. (5), the area function must be everywhere differentiable. Figure 4 illustrates the input impedances [cf. (17)] corresponding to the pipe constrictions of Fig. 3. For a constriction as near as possible to the common antinode at $x = 0$, the frequency shifts are found to be +30 Hz (6%), +76 Hz (5%), and +36 Hz (1%) at F_1, F_2, and F_3, respectively. Near the common node, however, for an otherwise-identical potential function, the shifts are found to be −109 Hz (22%), −214 Hz (14%), and −203 Hz (8%). Just anterior to the midpoint of the duct, furthermore, shifts of −70 Hz (14%), +24 Hz (2%), and −400 Hz (16%) have been found, now emphasizing the tuning of the third resonance. A finite difference solution of Eq. (4)

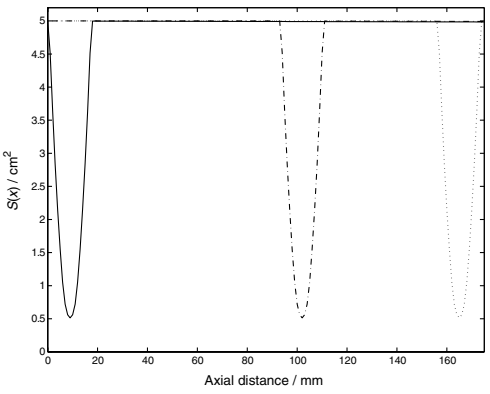

FIG. 3. Pipe constrictions corresponding to the potential function configuration of Fig. 2: localized at back of duct (solid line); near midpoint (dash-dotted line) ($a = 94$, $b = 95$, $c = 100$, $d = 106$, $e = 111$, and $f = 112$ mm); and at front of duct (dotted line).

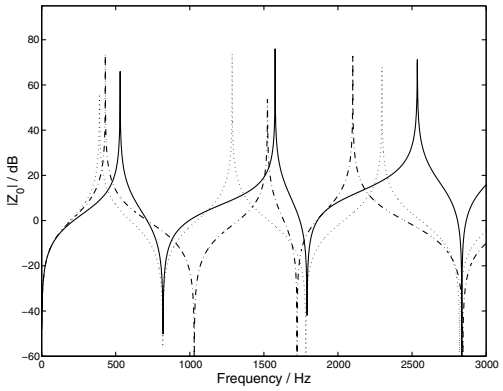

FIG. 4. Input impedance, Z_0, due to compound potential function corresponding to Figs. 2 and 3: at back (solid line), near midpoint (dash-dotted line), and at front of duct (dotted line).

confirmed these analytic results to better than 1% accuracy.

In light of the previous discussion, it is evident that the frequency shifts reflect the weighting of the potential function according to nominal standing wave pattern: Fig. 1 shows that there is a relatively broad region near the common antinode in which all sections of the compound potential function of Fig. 2 will be weighted equally but that, near the common node, it will be weighted more heavily towards the well on the antinode side. Thus, the net effect at an antinode is that of a barrier, and the frequency is raised; the net effect at a node is that of a well and a lowering of frequency. Near the duct midpoint, the weighing effects of the standing wave become more complex and depend on the width of the compound potential in relation to a quarter wavelength: for the third resonance, for example, centering just beyond the midpoint shifts the barrier section onto a node, isolating the wells and yielding a large negative frequency shift.

Recalling that the Rayleigh theory predicts equal but opposite effects at node and antinode, it is interesting to compare these analytic and finite difference results with those of the standard method, obtained by numerical solution of Eq. (1) for the three area functions of Fig. 3. For a constriction near the common antinode, the frequency shifts yielded by the standard method were found to be +25 Hz (5%), +66 Hz (4%), and +80 Hz (3%), respectively, in agreement to better than 2% with the wave-mechanical results. Near the duct midpoint, the standard method predicts shifts of −10 Hz (2%), +51 Hz (3%), and −145 Hz (6%), an underestimation of 12% in the first eigenvalue shift. Near the common node, where the theoretical predictions most strongly diverge, the standard shifts are found to be −31 Hz (6%), −81 Hz (5%), and −99 Hz (4%), and the underestimation in the first eigenvalue shift is 16%.

Conclusions.—In contrast to classical assumptions, it has been shown that it is precisely the small perturbations in potential energy, through dispersions due to the potential function, $U(x)$, that define correctly the eigenvalue shifts due to a change in the cross section of an acoustical duct. A Green's function analysis of piecewise-constant potential functions has shown that such a perturbation exactly at the nodes of the nominal eigenfunctions of a uniform pipe produces no shift in the respective eigenvalues. In the case of simple piecewise-constant potentials, maximal eigenvalue shifts are found at the antinodes. For a compound potential function, although there is no perturbation exactly at a node, shifts can be obtained *near* the node that differ in sign from those produced near an antinode, due to the weighting of the separate curvatures by the standing wave pattern. Just beyond the midpoint of the duct, compound potential functions can be used to produce large shifts in the first and third resonances, while leaving the second largely unperturbed. The theory presented here defines an accurate method of resonance analysis, with applications in engineering and musical acoustics, and the compact parametrization of speech.

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